On the Vacuum Energy in Expanding Space-Times*

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Abstract

If there is a shortest length in nature, for example at the Planck scale of $10^{-35}m$, then the cosmic expansion should continually create new comoving modes. A priori, each of the new modes comes with its own vacuum energy, which could contribute to the cosmological constant. I discuss possible mathematical models for a shortest length and an explicit model for a corresponding mode-generating mechanism.

1 Introduction

Theoretical evidence points towards the existence of a finite shortest length in nature. In particular, general relativity and quantum theory together indicate that the concept of distance loses operational meaning at the Planck scale: In any 'microscope', test particles aimed to resolve a region of Planckian size ($\approx 10^{-35}m$) would possess sufficient momentum uncertainty to significantly randomly curve and thereby disturb the region in space that they were meant to resolve.

The existence of a finite shortest length in nature could have cosmological implications. This is because in an expanding universe the independent modes are comoving modes, i.e. modes whose wavelength expands with the universe. Therefore, if there is a finite minimum length in nature, then new comoving modes of initially Planckian wave length must be created continually. A priori, each new mode comes with its own vacuum energy - which could contribute to the cosmological 'constant', and which could also be related to the inflaton potential.

During inflation, in particular, the new modes expand only about four or five orders of magnitude before their dynamics freezes upon crossing the

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Hubble horizon [1]. After such a brief expansion period the new modes could still possess properties that can be traced back to Planck scale physics. Therefore, relics of Planck scale physics which were frozen at horizon crossing could later have contributed to the seeding of density perturbations and may be observable in the CMB and in the subsequent structure formation. A signature of Planck scale effects could exist, for example, in contributions to the scalar/tensor ratio [2]. Crucial in this context is the curl component of the polarization spectrum of the CMB, which is likely to be measurable with MAP, Planck or a later satellite-based telescope.

2 Models of space-time with an ultraviolet cutoff

Any concrete candidate model of the short-distance structure of space-time must explain how new comoving modes continually arise and should therefore imply potentially measurable predictions for the cosmological constant, the inflaton potential and the seeding of structure formation.

A straightforward possibility is to model space-time as being discrete. It is still unclear, however, how space-time's continuity at large scales could emerge from a lattice description, in particular, as space-time is expanding. It is tempting, therefore, to speculate that a quantum gravity theory such as M-theory, see [3], or a foam theory, see [4], might eventually reveal the structure of space-time as being in between discrete and continuous, perhaps such as to combine the ultraviolet finiteness of lattices with continuous symmetry properties of manifolds.

Concretely, let us begin by asking whether it is at all possible that the points of space or of space-time form a set whose cardinality is in between discrete (i.e. countable) and continuous. The answer is a qualified no:

Cantor already conjectured (tragically) that there is no set whose cardinality lies in between discrete and continuous. In the year 1900, this so-called continuum hypothesis (CH) was listed by Hilbert as the first in his list of 23 problems for the 20th century. By the middle of the 20th century the problem's subtle solution was found by Gödel and Cohen. They showed that, within standard set theory, CH can neither be proved nor disproved.

This result means that on the one hand we could enforce the existence of sets of intermediate cardinality by simply claiming their existence in a new axiom for set theory. On the other hand, however, the answer of Cohen and Gödel implies that it would be impossible to explicitly construct any such set within standard set theory, since otherwise CH could be disproved. This also means that any *explicit* set of space-time points can only be of discrete

or continuous cardinality (theoretically, sets of space-time points could also be of higher than continuous cardinality, but it appears safe to discard this possibility since it would imply more rather than fewer ultraviolet divergencies).

We had asked whether there is a possibility by which nature might combine the ultraviolet finiteness of lattices with the continuous symmetry properties of manifolds. There still is a possibility.

3 Finiteness of the density of degrees of freedom

Let us recall that physical theories are formulated not directly in terms of points in space or in space-time but rather in terms of the functions over the set of points. This suggests a whole new class of mathematical models for a finite minimum length, which might beautifully combine the ultraviolet finiteness of lattices with the continuous symmetry properties of manifolds:

As first proposed in [5], fields in space-time could be functions over a continuous manifold as usual, while, crucially, the class of fields that can occur is such that if a field is sampled only at discrete points then its amplitudes can already be reconstructed at *all* points in the manifold - if the sampling points are spaced densely enough. The maximum average sample spacing which allows one to reconstruct the continuous field from discrete samples could be on the order of the Planck scale.

Since any one of all sufficiently tightly spaced lattices would allow reconstruction, no particular lattice would be preferred. Thus, all the continuity and symmetry properties of the manifold would be preserved. The physical theory could be written, equivalently, either as living on a continuous manifold, thereby displaying all its external symmetries, or as living on any one of the sampling lattices of sufficiently small average spacing, thereby displaying its ultraviolet finiteness. Physical fields, while being continuous or even differentiable, would possess only a finite density of degrees of freedom.

The mathematics of classes of functions which can be reconstructed from discrete samples is well-known, namely as *sampling theory,* in the information theory community, where it plays a central role in the theory of sources and channels of continuous information as developed by Shannon, see [6].

4 Sampling theory

The simplest example in sampling theory is the Shannon sampling theorem: Choose a frequency ω_{max}. Consider the class $B_{\omega_{max}}$ of continuous func-

tions f whose frequency content is limited to the interval $(-\omega_{max}, \omega_{max})$, i.e. for which: $\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} = 0$ whenever $|\omega| \geq \omega_{max}$. If the amplitudes $f(x_n)$ of such a function are known at equidistantly spaced discrete values $\{x_n\}$ whose spacing is π/ω_{max} or smaller, then the function's amplitudes $f(x)$ can be reconstructed for all x. The reconstruction formula is:

$$f(x) = \sum_{n=-\infty}^{\infty} f(x_n) \frac{\sin[(x - x_n)\omega_{max}]}{(x - x_n)\omega_{max}} \tag{1}$$

For the proof, note that $\tilde{f}$ has compact support and can therefore be represented both as a Fourier series of the $f(x_n)$ and as a Fourier transform of $f(x)$. The theorem is in ubiquitous use for example in digital audio and video. Sampling theory, see [7], studies generalizations of the theorem for various different classes of functions, for non-equidistant sampling, for multi-variable functions and it investigates the effect of noise, which could be quantum fluctuations in our case. Due to its engineering origin, sampling theory for generic pseudo- Riemannian manifolds is still virtually undeveloped, but should be of great interest. As was shown in [5], generalized sampling theorems automatically arise from stringy uncertainty relations, namely whenever there is a finite minimum position uncertainty Δx_{min}, as e.g. in uncertainty relations of the type: $\Delta x \Delta p \geq \frac{\hbar}{2}(1 + \beta(\Delta p)^2 + ...)$, see [8]. A few technical remarks: the underlying mathematics is that of symmetric non self-adjoint operators. Through a theorem of Naimark, unsharp variables of POVM type arise as special cases.

5 Application in Cosmology

It is possible to obtain explicit candidate models of quantum field theory close to the Planck scale by implementing a minimum length in nature as a sampling theoretical maximum information density. To this end, in [9], the above mentioned stringy uncertainty relations were implemented through their corresponding commutation relations. Physical fields then automatically become reconstructible everywhere from their samples on any lattice of spacing smaller than the Planck length.

The calculations were done for flat expanding background space-times, such as de Sitter space. As expected, comoving modes are continually being created. The modes' wave equations are still of the form $\Phi'' + m(\eta)\Phi' + n(\eta)\phi = 0$ but each mode now possesses of course a starting time η_0. At a given mode's starting time η_0 the coefficient functions $m(\eta)$ and $n(\eta)$ are singular. This singularity has not yet been fully understood. However, since

it determines the initial conditions and the vacuum its understanding will be crucial for predicting the size and signature of potentially observable effects. For the Hamiltonian, mode generation happens by continually adding the new modes' $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$ operators:

$$H = \int_{\vec{k}^2 < \frac{a(\eta)^2}{l_{Pl}^2}} d^3k \quad \left(\Phi'_{\vec{k}}\Pi_{\vec{k}} - \mathcal{L}_{\vec{k}}\right)$$

Here, $a(\eta)$ is the scale factor. Clearly, there is a continual production of vacuum energy if it is not artificially normal ordered away. Its size and properties are currently being investigated. It should be very interesting to apply sampling theory also to holographic bounds on the area density of information.

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